Kuwait University — Math 102 — Date: Jan 6, 2001
Dept of Math and Comp Sci — Final Exam — Duration: 2 hrs

Answer all of the following questions. Each (sub)question is worth 4 points.
Calculators, pagers and mobile telephones are NOT allowed.

1. Evaluate each limit.

(i) $\lim_{x\to 0^+} (x^2 + 2^x)^{\cot x}$ (ii) $\lim_{x\to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1}\right)$

2. Evaluate each integral.

(i) $\int \frac{dx}{\sqrt{x}(4\sqrt{x} + x^{1/3})}$ (ii) $\int \frac{2x+1}{\sqrt{3+2x-x^2}}\,dx$

(iii) $\int (\sin^3 x)\ln(\cos x)\,dx$ (iv) $\int \frac{dx}{\sqrt{x}(1-\cos\sqrt{x})}$

3. Determine whether the improper integral converges or diverges. If it converges, find its limit.

$$\int_0^\infty \frac{dx}{e^{-x}+1}$$

4. Sketch the graph of $r = \sin 2\theta$, and find the area of the region enclosed.

5. Show that the line

$$l: \quad x = 1-2t, \quad y = 2+3t, \quad z = 3-t$$

is parallel to the line of intersection of the two planes

$$x+y+z-1=0 \quad \text{and} \quad 2x+y-z-3=0.$$

6. Find an equation of the plane through the points
$P(3,2,1)$, $Q(-1,1,-2)$, and $R(3,-4,1)$.

Final Exam (Solutions)

$\lim_{x\to 0^+} \cot x \ln(x^2+2^x) = \lim_{x\to 0^+} \frac{\ln(x^2+2^x)}{\tan x} \overset{L'H}{=} \lim_{x\to 0^+} \frac{\frac{2x+2^x\ln 2}{x^2+2^x}}{\sec^2 x} = \ln 2.$

$\lim_{x\to 0^+} (x^2+2^x)^{\cot x} = e^{\ln 2} = 2$; (ii) $\lim_{x\to 1^+}\left(\frac{1}{\ln x} - \frac{1}{x-1}\right) = \lim_{x\to 1^+} \frac{x-1-\ln x}{(x-1)\ln x} \overset{L'H}{=}$

$\lim_{x\to 1^+} \frac{x-1}{x\ln x + x - 1} \overset{L'H}{=} \lim_{x\to 1^+} \frac{1}{\ln x + 2} = \frac{1}{2}.$

i) $\sqrt[6]{x} = u$, $x = u^6$, $dx = 6u^5 du$, $\sqrt[3]{x} = u^2$, $\sqrt{x} = u^3 \implies$

$I = 6\int \frac{u^5 du}{u^3(4u^3+u^2)} = 6\int \frac{du}{4u+1} = \frac{3}{2}\ln|4\sqrt[6]{x}+1| + C.$

ii) $I = \int \frac{2x+1}{\sqrt{4-(x-1)^2}}dx = \int \frac{2t+3}{\sqrt{4-t^2}}dt = -2\sqrt{3+2x-x^2} + 3\sin^{-1}\frac{x-1}{2} + C$

$(x-1=t,\ dx=dt)$

iii) $I = \int (1-\cos^2 x)\ln(\cos x)\sin x\, dx = \int (t^2-1)\ln t\, dt \overset{\text{(by parts)}}{=} \left(\frac{t^3}{3}-t\right)\ln t$

$(\cos x = t,\ -\sin x\, dx = dt)\left(\begin{matrix} u=\ln t, & dv=(t^2-1)dt \\ du=\frac{dt}{t}, & v=\frac{t^3}{3}-t\end{matrix}\right)$

$-\int\left(\frac{t^2}{3}-1\right)dt = \left(\frac{\cos^3 x}{3} - \cos x\right)\ln(\cos x) - \frac{\cos^3 x}{9} + \cos x + C.$

v) $\sqrt{x} = t \quad \frac{dx}{\sqrt{x}} = 2dt \implies I = 2\int \frac{dt}{1-\cos t} = 2\int \frac{\frac{2du}{1+u^2}}{1-\frac{1-u^2}{1+u^2}} = 2\int \frac{du}{u^2} = -\frac{2}{\tan\frac{\sqrt{x}}{2}} + C.$

$(\tan\frac{t}{2} = u)$

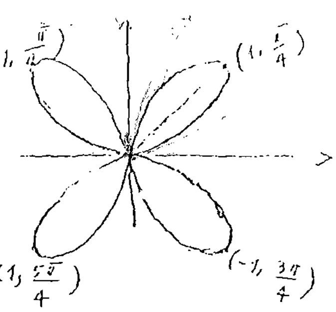
or: $I = 2\int \frac{dt}{1-\cos t} = 2\int \frac{1+\cos t}{\sin^2 t}dt = -2(\cot t + \csc t) + C = -2(\cot\sqrt{x} + \csc\sqrt{x}) + C.$

Compute: $\lim_{t\to\infty}\int_0^t \frac{1}{e^{-x}+1}dx = \lim_{t\to\infty}\int_0^t \frac{e^x}{1+e^x}dx = \lim_{t\to\infty}\left[\ln(1+e^x)\right]_0^t = \infty.$

Thus the integral diverges.

$\text{Area}(R) = 4\left[\frac{1}{2}\int_0^{\pi/2}\sin^2 2\theta\, d\theta\right] = \int_0^{\pi/2}(1-\cos 4\theta)d\theta$

$= \left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi}{2}.$



The direction of the first line is given by the vector $\vec{u} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -2\vec{i} + 3\vec{j} - \vec{k}$ which coincides with the vector that gives the direction of ℓ.

$\overrightarrow{PQ} = \langle -4, -1, -3 \rangle$,

$\overrightarrow{PR} = \langle 0, -6, 0 \rangle \implies \vec{n} = \overrightarrow{PQ}\times\overrightarrow{PR} = -6(3\vec{i} - 4\vec{k})$

The equation of the plane: $3(x-3) - 4(z-1) = 0 \iff 3x - 4z - 5 = 0$